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Central and non-central extensions of multi-graded Lie algebras

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Abstract. We construct non-central extensions of the algebras of vector fields and gauge transformations in N-dimensional space. In one dimension they reduce to the ordinary central extensions occurring in Virasoro and Kac–Moody algebras.

1. Introduction

We recall that the Virasoro algebra [1],

$$[L(m), L(n)] = (n-m)L(m+n) - \frac{c}{12}(m^3 - m)\delta(m+n) \tag{1.1}$$

where $m, n \in \mathbb{Z}$ is the universal central extension of Vect(1), the algebra of vector fields in one dimensions. Recall also that the Kac–Moody algebra is the universal central extension of Map$(1, g)$, the algebra of maps from the circle to the finite-dimensional Lie algebra g. It has the brackets

$$[J^a(m), J^b(n)] = f^{abc}J^c(m+n) + \delta^{ab}km\delta(m+n) \tag{1.2}$$

where a, b are g indices, f^{abc} are the totally antisymmetric structure constants and δ^{ab} is the Killing metric of g. Due to this metric there is no need to distinguish between upper and lower g indices.

It is evident that these algebras admit the following representations when the extensions vanish:

$$L(m) = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}mx}\partial \qquad J^a(m) = \mathrm{e}^{\mathrm{i}mx}\,M^a \tag{1.3}$$

where M^a are matrices in some finite-dimensional representation of g. We can immediately write down the corresponding generalization of these representations to N-dimensional vector fields and maps.

$$L^\mu(m) = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}m\cdot x}\partial^\mu \qquad J^a(m) = \mathrm{e}^{\mathrm{i}m\cdot x}\,M^a \tag{1.4}$$

where $x_\mu = (x_1, \ldots, x_N)$ and $m^\mu = (m^1, \ldots, m^N)$ are N-dimensional vectors and $m\cdot x \equiv m^\sigma x_\sigma$ denotes the scalar product of m and x. Since there is no metric on the base

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manifold, the difference between upper and lower indices is significant. We are thus led to define the semi-direct product Vect(N)⋉Map(N, g) by

$$\begin{aligned}
&[L^\mu(m), L^\nu(n)] = n^\mu L^\nu(m+n) - m^\nu L^\mu(m+n) \\
&[L^\mu(m), J^b(n)] = n^\mu J^b(m+n) \\
&[J^a(m), J^b(n)] = f^{abc} J^c(m+n).
\end{aligned} \tag{1.5}$$

Strictly speaking, this is the correct algebra on the N-dimensional torus only, but the local considerations apply to arbitrary manifolds.

Because the central extension plays a fundamental role in one dimension, it is natural to ask if any generalization to higher dimensions is possible. The purpose of this paper is to show that there is a non-central extension which precisely reduces to the central one in one dimension. We will also construct representations of the extended algebras. However, from the construction it will be rather clear that these non-central extensions are quite uninteresting. The point is that central extensions in one dimension arise naturally from normal ordering in Fock modules, whereas the higher-dimensional generalizations have nothing to do with Fock modules.

We have previously obtained some results in this direction [2], but the present work gives a more complete treatment of the problem. Moreover, we give here a geometrical description of the extensions which explains their appearance. The connection with the Hamiltonian formulation of gauge anomalies is also discussed.

2. Central extensions

We first note that Map(N, g) has a central extension which is an immediate generalization of the one-dimensional case. The brackets are

$$[J^a(m), J^b(n)] = f^{abc} J^c(m+n) + \delta^{ab} k \cdot m \delta(m+n) \tag{2.1}$$

where k_σ is central and $k \cdot m \equiv k_\sigma m^\sigma$. The proof consists of checking the Jacobi identities.

$$\begin{aligned}
&[J^a(m), [J^b[n], J^c(s)]] + \ldots \\
&\quad = [J^a(m), f^{bcd} J^c(n+s)] + \ldots \\
&\quad = \mathrm{reg} + f^{bcd} \delta^{ad} k \cdot m \delta(m+n+s) + \ldots \\
&\quad = \mathrm{reg} + f^{abc} k \cdot (m+n+s) \delta(m+n+s) = 0
\end{aligned} \tag{2.2}$$

where reg stand for regular terms, proportional to $J^d(m+n+s)$, and the dots indicate the two cyclic permutations.

However, this central extension is not meaningful because it is not possible to generalize it to an extension of Vect(N)⋉Map(N, g). Because the extension does not admit an intertwining action of Vect(N) it is an artefact of the choice of coordinate system on the base manifold and not an intrinsic object. To prove this statement is suffices to limit our attention to g Abelian; this is the ordinary function algebra in N dimensions. The only identity which could possibly go wrong is LLJ, and it does:

$$\begin{aligned}
&[L^\mu(m), [J(n), J(s)]] + [J(s), [L^\mu(m), J(n)]] + [J(n), [J(s), L^\mu(m)]] \\
&\quad = [J(s), n^\mu J(m+n)] - [J(n), s^\mu J(m+s)] \\
&\quad = (n^\mu k \cdot s - s^\mu k \cdot n) \delta(m+n+s).
\end{aligned} \tag{2.3}$$

In one dimension, this is identically zero because $nks - skn = 0$ for ordinary commuting numbers. When $N > 1$ the expression vanishes only if $k \cdot n = n^\mu$ and similarly for s. However, this equation cannot hold simultaneously for every $\mu = 1, \ldots, N$ and therefore the *LJJ* Jacobi identity fails.

The same argument also applies to the Poisson bracket algebra and the Moyal algebra of quantum deformed functions, because the ordinary function algebra is a special case of these algebras, and hence they do not admit meaningful central extensions either. This point has sometimes been overlooked in the literature [3].

The proof that Vect(N) does not admit any central extension has occurred elsewhere [4, 5], and therefore we only sketch the argument. Let $L(k) = \alpha \cdot L(km)/\alpha \cdot m$, $k, l \in \mathbb{Z}$, $m \in \Lambda$, and α_μ is a fixed vector. Then it is clear that $L(k)$ obeys Vect(1), whose unique central extension is the Virasoro algebra. Because this is true for arbitrary m and α_μ, any non-trivial extension of Vect(N) must be cubic. The most general cubic ansatz fails the Jacobi identities, except when $N = 1$.

3. Non-central extensions

In this section we reformulate the central extensions of the $N = 1$ algebras in a fashion which admits a generalization to higher dimensions. The Kronecker delta is defined up to a constant factor by the constraint $n\delta(n) = 0$, because either $\delta(n) = 0$ or $n = 0$. This condition is evidently preserved by the action of Vect(N)$\ltimes$Map(N, g) because both n and $\delta(n)$ transform trivially; hence the condition is meaningful.

To formulate our result we need some standard results from differential geometry, which were adapted to the present formalism in [5].

A tensor field $\mathcal{T}^p_q(\lambda)$ is a Vect(N) module with basis $\{\phi^{\sigma_1 \ldots \sigma_p}_{\tau_1 \ldots \tau_q}(n)\}_{n \in \Lambda}$. The action of Vect($N$) is given by

$$
\begin{aligned}
L^\mu(m)\phi^{\sigma_1 \ldots \sigma_p}_{\tau_1 \ldots \tau_q}(n) \\
&= ((1-\lambda)m^\mu + n^\mu)\phi^{\sigma_1 \ldots \sigma_p}_{\tau_1 \ldots \tau_q}(m+n) \\
&\quad - \sum_{i=1}^{p} m^{\sigma_i}\phi^{\sigma_1 \ldots \mu \ldots \sigma_p}_{\tau_1 \ldots \tau_q}(m+n) + \sum_{j=1}^{q} \delta^\mu_{\tau_j} m^\nu \phi^{\sigma_1 \ldots \sigma_p}_{\tau_1 \ldots \nu \ldots \tau_q}(m+n).
\end{aligned}
\tag{3.1}
$$

There is a $\mathcal{T}^0_p(1)$ submodule Ω_p consisting of tensor fields $S_{\tau_1 \ldots \tau_q}(n)$ with p totally antisymmetric lower indices. The elements in Ω_p are called p-chains.

There is a module homomorphism which is dual to the exterior derivative.

$$
\begin{aligned}
\delta_p: \qquad & \Omega_p \to \Omega_{p-1} \\
& S_{\nu_1 \ldots \nu_p}(n) \mapsto (\delta_p S)_{\nu_1 \ldots \nu_{p-1}}(n) = n^{\nu_p} S_{\nu_1 \ldots \nu_{p-1}\nu_p}(n).
\end{aligned}
\tag{3.2}
$$

Further, $\delta_{p-1}\delta_p = 0$. The Ω_p submodules ker δ_p and im δ_{p+1} are identified as closed and exact p-chains, respectively. We henceforth suppress the index p on this homomorphism.

Let us present the proof that (3.2) defines a homomorphism in the case of one-chains, which will be our main concern in this paper.

$$
\begin{aligned}
L^\mu(m)(\delta S)n &= L^\mu(m) n^\nu S_\nu(n) \\
&= n^\nu(n^\mu S_\nu(m+n) + \delta^\mu_\nu m \cdot S(m+n)) \\
&= n^\mu(m+n) \cdot S(m+n) = n^\mu(\delta S)(m+n).
\end{aligned}
\tag{3.3}
$$

The existence of the homomophism (3.2) makes it possible to consistently require that $\delta S=0$ without demanding that $S_\nu\equiv 0$. In the special case that $N=1$ we can solve this constraint for $S(n)$: $nS(n)=0$ implies that $S(n)=0$ except when $n=0$. The unique solution is thus $S(n)=\delta(n)$, up to a constant factor. When $N=1$ we can still consistently impose the condition $n\cdot S(n)=0$, but the solution is now non-trivial.

If we replace $\delta(m+n)$ in (1.1-2) by a closed one-chain $S(m+n)$, we obtain an expression which immediately generalizes to arbitrary N. The following non-central extension of $\mathrm{Vect}(N)\ltimes\mathrm{Map}(N,g)$ is a Lie algebra

$$
\begin{aligned}
&[L^\mu(m), L^\nu(n)]=n^\mu L^\nu(m+n)-m^\nu L^\mu(m+n)-\frac{c}{24}m^\mu n^\nu(m-n)\cdot S(m+n)\\
&[L^\mu(m), J^b(n)]=n^\mu J^b(m+n)\\
&[J^a(m), J^b(n)]=f^{abc}J^c(m+n)+\delta^{ab}\frac{k}{2}(m-n)\cdot S(m+n)\\
&[L^\mu(m), S_\nu(n)]=n^\mu S_\nu(m+n)+\delta^\mu_\nu m\cdot S(m+n)\\
&[J^a(m), S_\nu(n)]=[S_\mu(m), S_\nu(n)]=0
\end{aligned}
\tag{3.4}
$$

provided that $S_\mu(m)$ is subject to the constraint $n\cdot S(n)=0$.

The extension of $\mathrm{Vect}(N)$, without the constraint, was reported in [2]. To prove that (3.4) is a Lie algebra, we note that the only non-trivial Jacobi identities are LLL, LJJ and JJJ, because the others are either trivially true (because $[J,S]=[S,S]=0$), or follow from the representation condition for tensor fields. LLL was proved in [2] and the proof of JJJ is identical to the central case (2.2), since $(m+n+s)\cdot S(m+n+s)=0$. Remains to check the Jacobi identity which goes wrong for the central extension.

$$
\begin{aligned}
&[L^\mu(m),[J^a(n),J^b(s)]]+[J^b(s),[L^\mu(m),J^a(n)]]+[J^a(n),[J^b(s),L^\mu(m)]]\\
&\quad=\left[L^\mu(m), f^{abc}J^c(n+s)+\delta^{ab}\frac{k}{2}(n-s)\cdot S(n+s)\right]\\
&\qquad+[J^b(s), n^\mu J^a(m+n)]-[J^a(n), s^\mu J^b(m+s)]\\
&\quad=\mathrm{reg}+\delta^{ab}\frac{k}{2}((n^\mu+s^\mu)(n-s)\cdot S+(n^\mu-s^\mu)m\cdot S\\
&\qquad+n^\mu(s-m-n)\cdot S-s^\mu(n-m-s)\cdot S)=0
\end{aligned}
\tag{3.5}
$$

where reg stands for regular terms, proportional to J^c.

To understand the nature of these equations, it is helpful to reformulate them in a position-space basis. Let

$$
f_\mu(x)=\sum_m \mathrm{e}^{-\mathrm{i}m\cdot x}f_\mu(m)\qquad L(f)=\sum_m f_\mu(m)\mathrm{i}L^\mu(-m).
\tag{3.6}
$$

Further, we can identify the one-chain with a line integral over some curve C:

$$
\begin{aligned}
&S_\nu(n)=\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^{\mathrm{i}n\cdot x}\,\mathrm{d}x_\nu\\
&n\cdot S(n)=\frac{-1}{2\pi}\int_C \partial^\nu(\mathrm{e}^{\mathrm{i}n\cdot x})\,\mathrm{d}x_\nu=\frac{-1}{2\pi}\mathrm{e}^{\mathrm{i}n\cdot x}\Big|
\end{aligned}
\tag{3.7}
$$

and this expression vanishes if the curve C is closed. The normalization is such that $S(0)=1$ on the unit circle. This motivates the name closed one-forms for the space $\ker \delta_1$:

$$[L(f), L(g)]$$

$$= -\sum_{mn} f_\mu(m) g_\nu(n)\Big(-n^\mu L^\nu(-m-n) + m^\nu L^\mu(-m-n)$$

$$+\frac{c}{48\pi \mathrm{i}} m^\mu n^\nu (m^\sigma - n^\sigma)\int_C \mathrm{e}^{-\mathrm{i}(m+n)\cdot x}\, \mathrm{d}x_\sigma\Big)$$

$$= L(f\cdot\partial g - g\cdot\partial f) + \frac{c}{48\pi}\int_C (\partial\cdot g\partial^\sigma\partial\cdot f - \partial\cdot f\partial^\sigma\partial\cdot g)\, \mathrm{d}x_\sigma. \tag{3.8}$$

By an integration by parts, the extension becomes

$$\frac{c}{24\pi}\int_C \partial\cdot g\partial^\sigma\partial\cdot f\, \mathrm{d}x_\sigma. \tag{3.9}$$

Similarly, if we introduce

$$J(\phi) = \sum_m \phi^a(m) J^a(-m) \tag{3.10}$$

the extension of $\mathrm{Map}(N, g)$ takes the form

$$[J(\phi), J(\psi)]$$

$$= \sum_{mn} \phi^a(m)\psi^b(n)\Big(f^{abc}J^c(-m-n) + \delta^{ab}\frac{k}{2}(n-m)\cdot S(-m-n)\Big)$$

$$= J([\phi, \psi]) - \frac{k}{4\pi}\int_C (\langle\partial^\sigma, \phi, \psi\rangle - \langle\phi, \partial^\sigma\psi\rangle dx_\sigma) \tag{3.11}$$

where $\langle\phi, \psi\rangle = \phi^a\psi^a$ is the Killing form on g.

We are now in the position to understand why the extensions are non-central in more than one dimension. On the circle there is a unique closed curve along which to integrate: the entire circle. This curve does not change under arbitrary one-dimensional diffeomorphisms, although the points on the curve move relative to each other. The latter effect can be compensated by a shift in the integration variable. However, a curve embedded in $N>1$ dimensions changes form under N-dimensional diffeomorphisms, which generically have a component perpendicular to the curve. If the curve is open it transforms non-trivially already in one dimension, because and end-points move.

A trivial way to obtain a closed one-chain is clearly to take an exact one. This is achieved by substituting $S_\mu(m) = m^\sigma R_{\mu\sigma}(m)$, $R_{\mu\nu}\in\Omega_2$, i.e.

$$\begin{aligned}&[L^\mu(m), R_{\sigma\tau}(n)] = n^\mu R_{\sigma\tau}(m+n) + \delta^\mu_\sigma m^\rho R_{\rho\tau}(m+n) + \delta^\mu_\tau m^\rho R_{\rho\sigma}(m+n)\\ &R_{\mu\nu}(m) = -R_{\nu\mu}(m)\end{aligned} \tag{3.12}$$

into (3.4). The extensions then acquire the forms

$$\begin{aligned}&-\frac{c}{12} m^\mu n^\nu m^\sigma n^\tau R_{\sigma\tau}(m+n)\\ &\delta^{ab} k m^\sigma n^\tau R_{\sigma\tau}(m+n)\end{aligned} \tag{3.13}$$

respectively. These extension were also mentioned in [2]. In position space, the corresponding result is

$$[L(f), L(g)] = L(f\cdot\partial g - g\cdot\partial f) + \frac{c}{24\pi}\int\!\!\int_S \partial^\sigma\partial\cdot f\partial^\tau\partial\cdot g\,\mathrm{d}x_\sigma\,\mathrm{d}x_\tau$$
$$[J(\phi), J(\psi)] = J([\phi,\psi]) - \frac{k}{2\pi}\int\!\!\int_S \langle\partial^\sigma\phi, \partial^\tau\psi\rangle\,\mathrm{d}x_\sigma\,\mathrm{d}x_\tau \tag{3.14}$$

where S is some two-dimensional surface.

We remark that if more variables are introduced, it is no longer necessary to limit ourselves to line and surface integrals. For example, consider the structure of the commutator anomaly of current algebra in three space dimensions [6]. It has the form

$$-\frac{1}{24\pi^2}\int\!\!\int\!\!\int_V \operatorname{tr}\partial^\sigma\phi\partial^\tau\psi A^\rho\,\mathrm{d}x_\rho\,\mathrm{d}x_\sigma\,\mathrm{d}x_\tau \tag{3.15}$$

where A^ρ is the gauge potential. We obtain an extension of $\mathrm{Map}(N, g)$ of the form

$$[J^a(m), J^b(n)] = f^{abc}J^c(m+n) + m^\mu n^\nu R^{ab}_{\mu\nu}(m+n)$$
$$[J^a(m), R^{bc}_{\mu\nu}(n)] = f^{abd}R^{dc}_{\mu\nu}(m+n) + f^{acd}R^{cd}_{\mu\nu}(m+n) \tag{3.16}$$

where

$$R^{ab}_{\mu\nu}(m) = -\frac{1}{24\pi^2}\int\!\!\int\!\!\int_V \mathrm{e}^{\mathrm{i}m\cdot x}\,d^{abc}A^{c\rho}\,\mathrm{d}x_\rho\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu \tag{3.17}$$

and $d^{abc} = \operatorname{tr}(\{M^a, M^b\}M^c)$.

4. Representations on bosons and fermions

We now present a method to construct a class of representations of the non-central extension of $\mathrm{Vect}(N)$.

Assume that $L^\mu_0(m)$ are generators of $\mathrm{Vect}(N)$, $X \in \mathcal{T}^0_0(1)$ and $Y_\nu \in \mathcal{T}^0_1(1)$ are representations of this algebra, and

$$[X(m), X(n)] = [X(m), Y_\nu(n)] = [Y_\mu(m), Y_\nu(n)] = 0. \tag{4.1}$$

Then

$$L^\mu(m) = L^\mu_0(m) + m^\mu(X(m) + m\cdot Y(m)) \tag{4.2}$$

also satisfies $\mathrm{Vect}(N)$. It actually suffices to prove this for $Y_\mu = 0$, due to the homomorphism δ_1 (3.2).

$$\begin{aligned}[L^\mu_0(m) + m^\mu X(m), L^\nu_0(n) + n^\nu X(n)] &= n^\mu L^\nu_0(m+n) + n^\nu n^\mu X(m+n) - m \leftrightarrow n\\ &= n^\mu L^\nu_0(m+n) - m^\nu n^\mu X(m+n) - m \leftrightarrow n.\end{aligned} \tag{4.3}$$

which is the claimed result because $m^\nu n^\mu X(m+n)$ is invariant under interchange of m and n (and, simultaneously, μ and ν). This construction is intimately related to the existence of a connection. In fact, we can identify $Y_\mu = \bar{\Gamma}^\sigma_{\sigma\mu}$, where the latter element is conjugate to the Cristoffel symbols.

Let us relax the conditions (4.1) to

$$\begin{aligned}&[X(m), X(n)]=0\\&[X(m), Y_\nu(n)]=S_\nu(m+n)\\&[Y_\mu(m), Y_\nu(n)]=R_{\mu\nu}(m+n)\end{aligned}\tag{4.4}$$

where the new operators S_ν and $R_{\mu\nu}$ have to be elements in the modules Ω_1 and Ω_2, respectively. This follows because the action of $L_0^\mu(m)$ on the LHS and RHS of the above equations must agree. The commutators of Vect(N) are now replaced by

$$\begin{aligned}&[L^\mu(m), L^\nu(n)]\\&\quad=n^\mu L^\nu(m+n)-m^\nu L^\mu(m+n)\\&\qquad+m^\mu n^\nu((n-m)\cdot S(m+n)+m^\sigma n^\tau R_{\sigma\tau}(m+n))\end{aligned}\tag{4.5}$$

which are the previous extensions of Vect(N) apart from a trivial rescaling.

It should be noted that this construction gives an independent proof that (4.5) is a Lie algebra. Namely, it suffices to check that the conditions (4.4) are consistent.

We now explain how (4.2) can be used to build representations in the enveloping algebra of canonical commutation and anticommutation relations. Following [5], we make the following set of definitions. Let $a_\mu(m)$ be a bosonic vector field and $\bar{a}^\mu(m)$ its canonical conjugate, subject to the conditions

$$\begin{aligned}&[a_\mu(m), \bar{a}^\nu(n)]=\delta_\mu^\nu\delta(m+n)\\&[a_\mu(m), a_\nu(n)]=[\bar{a}^\mu(m), \bar{a}^\nu(n)]=0.\end{aligned}\tag{4.6}$$

Moreover, let the fermionic scalar field $b(m)$ and its conjugate $\bar{b}(m)$ satisfy

$$\begin{aligned}&\{b(m), \bar{b}(n)\}=\delta_\mu^\nu\delta(m+n)\\&\{b(m), b(n)\}=\{\bar{b}(m), \bar{b}(n)\}=0.\end{aligned}\tag{4.7}$$

Define further the convolution of two fields by

$$ab(m)\equiv\sum_{s\in\Lambda}a(m-s)b(s)\tag{4.8}$$

and the derivative acts in momentum space as $\mathrm{i}\partial^\mu a(s)=s^\mu a(s)$.

With these preliminaries, we can now check that the following set of operators satisfies the conditions above.

$$\begin{aligned}&L_0^\mu(m)=-\bar{a}^\sigma\,\mathrm{i}\partial^\mu a_\sigma(m)-m^\sigma\bar{a}^\mu a_\sigma(m)-\bar{b}\,\mathrm{i}\partial^\mu b(m)\\&X(m)=-\bar{a}\cdot a(m)\equiv N_a(m)\\&Y_\mu(m)=-\bar{b}ba_\mu(m)\equiv N_b a_\mu(m)\end{aligned}\tag{4.9}$$

where the number operators satisfy

$$\begin{aligned}&[N_a(m), \bar{a}^\nu(n)]=-\bar{a}^\nu(m+n) &&[N_a(m), a_\nu(n)]=a_\nu(m+n)\\&[N_b(m), \bar{b}(n)]=-\bar{b}(m+n) &&[N_b(m), b(n)]=b(m+n).\end{aligned}\tag{4.10}$$

One now checks that

$$\begin{aligned}&[L_0^\mu(m), X(n)]=n^\mu X(m+n)\\&[L_0^\mu(m), Y_\nu(n)]=n^\mu Y_\nu(m+n)+\delta_\nu^\mu m\cdot Y(m+n)\\&[X(m), Y_\nu(n)]=N_b a_\nu(m+n)=Y_\nu(m+n)\equiv S_\nu(m+n)\\&[Y_\mu(m), Y_\nu(n)]=0.\end{aligned}\tag{4.11}$$

If we substitute these expressions into the commutators, we find that all brackets in (3.4) are satisfied.

Despite the representability of the non-central extensions, we doubt that they have any true significance. The reason is that the central extension of Vect(1) arises naturally when one studies Fock modules, by the normal ordering prescription. Normal ordering does not make sense in more than one dimension, because it would give rise to an infinite central extension rather than a non-central extension. In fact, the coefficient front of the central extension would be proportional to ∞^{N-1} in N dimensions [5].

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